by Ian Stewart

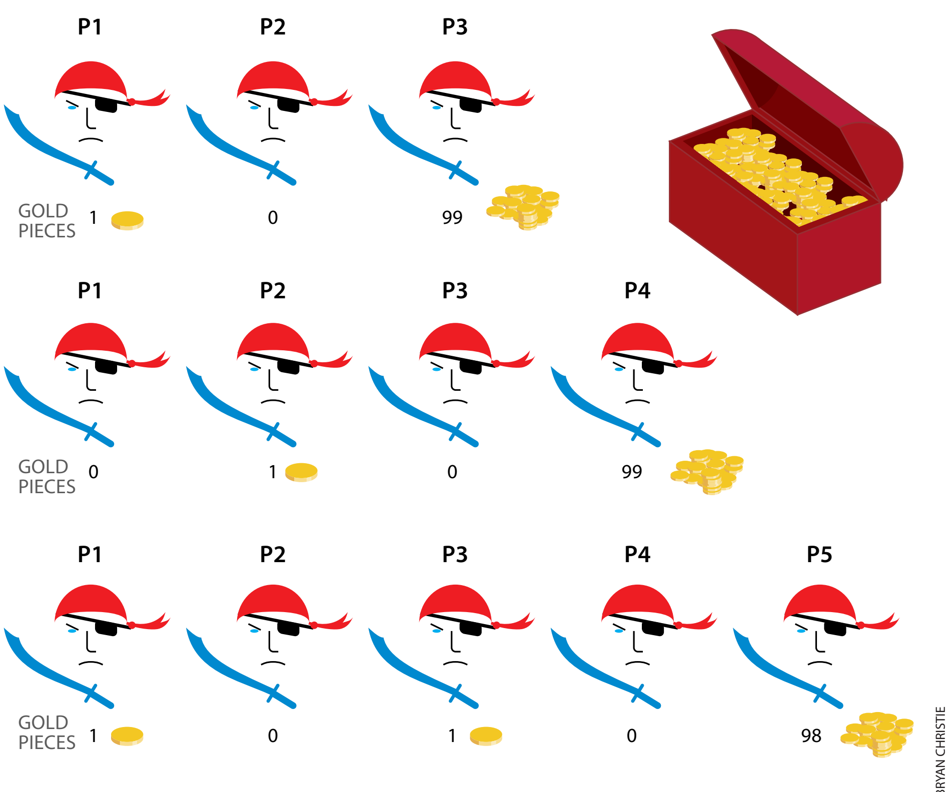
A Puzzle for Pirates

The logic of mathematics sometimes leads to apparently bizarre conclusions. The rule here is that if the logic doesn't have holes in it, the conclusions are sound, even if they conflict with your intuition. In September 1998 Stephen M. Omohundro of Palo Alto, Calif., sent me a puzzle that falls into exactly this category. The puzzle has been circulating for at least 10 years, but Omohundro came up with a variant in which the logic becomes surprisingly convoluted.

First, the original version of the puzzle. Ten pirates have gotten their hands on a hoard of 100 gold pieces and wish to divide the loot. They are democratic pirates, in their own way, and it is their custom to make such divisions in the following manner: The fiercest pirate makes a proposal about the division, and everybody votes on it, including the proposer. If 50 percent or more are in favor, the proposal passes and is implemented forthwith. Otherwise the proposer is thrown overboard, and the procedure is repeated with the next fiercest pirate.

All the pirates enjoy throwing one of their fellows overboard, but if given a choice they prefer cold, hard cash. They dislike being thrown overboard themselves. All pirates are rational and know that the other pirates are also rational. Moreover, no two pirates are equally fierce, so there is a precise pecking order—and it is known to them all. The gold pieces are indivisible, and arrangements to share pieces are not permitted, because no pirate trusts his fellows to stick to such an arrangement. It's every man for himself.

What proposal should the fiercest pirate make to get the most gold? For convenience, number the pirates in order of meekness, so that the least fierce is number 1, the next least fierce number 2 and so on. The fiercest pirate thus gets the biggest number, and proposals proceed in reverse order from the top down.



FIERCEST PIRATES
get the lion's share of gold when the loot is divided among groups of three, four or five pirates.

The secret to analyzing all such games of strategy is to work backward from the end. At the end, you know which decisions are good and which are bad. Having established that, you can transfer that knowledge to the next-to-last decision and so on. Working from the beginning, in the order in which the decisions are actually taken, doesn't get you very far. The reason is that strategic decisions are all about "What will the next person do if I do this?" so the decisions that follow yours are important. The ones that come before yours aren't, because you can't do anything about them anyway.

Bearing this in mind, the place to start is the point at which the game gets down to just two pirates, P1 and P2. The fiercest pirate at this point is P2, and his optimal decision is obvious: propose 100 gold pieces for himself and none for P1. His own vote is 50 percent of the total, so the proposal wins.

Now add in pirate P3. Pirate P1 knows—and P3 knows that he knows—that if P3's proposal is voted down, the game will proceed to the two-pirate stage and P1 will get nothing. So P1 will vote in favor of anything that P3 proposes, provided it yields him more than nothing. P3 therefore uses as little as possible of the gold to bribe P1, leading to the following proposal: 99 for P3, 0 for P2 and 1 for P1 [*see illustration at left*].

The strategy of P4 is similar. He needs 50 percent of the vote, so again he needs to bring exactly one other pirate on board. The minimum bribe he can use is one gold piece, and he can offer this to P2 because P2 will get nothing if P4's proposal fails and P3's is voted on. So P4's proposal is 99 for himself, 0 for P3, 1 for P2 and 0 for P1. The approach taken by P5 is slightly different: he needs to bribe *two* pirates to get a winning vote. The minimum bribe he can use is two gold pieces, and the unique way he can succeed with this bribe is to propose 98 gold pieces for himself, 0 for P4, 1 for P3, 0 for P2 and 1 for P1.

Possible Recipients of One Gold Piece

202 PIRATES

P1	P2	P3	P4	....	P197	P198	P199	P200	P201	P202
NO	YES	NO	YES	....	NO	YES	NO	YES	YES	NO

204 PIRATES

P1	P2	P3	P4	....	P197	P198	P199	P200	P201	P202	P203	P204
YES	NO	YES	NO	....	YES	NO	YES	NO	NO	YES	NO	NO

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IN LARGER GROUPS,
the fiercest pirate must bribe 100 of his fellows with one gold piece each.

The analysis proceeds in the same manner, with each proposal uniquely prescribed to give the proposer the maximum reward while also ensuring a favorable vote. Following this pattern, P10 will propose 96 gold pieces for himself, one gold piece for each of pirates P8, P6, P4 and P2, and none for the odd-numbered pirates. This allocation solves the 10-pirate version of the puzzle.

Omohundro's contribution is to ask the same question but with 500 pirates instead of 10 divvying up the 100 gold pieces. Clearly, the same pattern persists—for a while. In fact, it persists up to the 200th pirate. P200 will offer nothing to the odd-numbered pirates P1 through P199 and one gold piece to each of the even-numbered pirates P2 through P198, leaving one for himself. At first sight, the argument breaks down after P200, because P201 has run out of bribes. Yet P201 still has a vested interest in not being thrown overboard, so he can propose to take nothing himself and offer one gold piece to each of the odd-numbered pirates P1 through P199.

Pirate P202 also is forced to accept nothing—he must use the entire 100 gold pieces to bribe 100 pirates, and these recipients must be among those who would get nothing under P201's proposal. Because there are 101 such pirates, P202's proposal is no longer unique—there are 101 ways to distribute the bribes. The illustration above shows the 101 pirates who *might* get something from P202's proposal and the 101 pirates who would definitely get nothing.

Pirate P203 must obtain 102 favorable votes, including his own, and he clearly doesn't have enough cash available to bribe 101 of his fellow pirates. So P203 goes overboard no matter what he proposes. Even though P203 is destined to walk the plank, this doesn't mean that he plays no part in the proceedings. On the contrary, P204 now knows that P203's sole aim in life is to avoid having to propose a division of the spoils. So P204 can count on P203's vote, whatever P204 proposes. Now P204 just squeaks home: he can count on his own vote, P203's vote and 100 others from bribes of one gold coin each—102 votes in all, the necessary 50 percent. The recipients of the bribes must be among the 101 pirates who would definitely receive nothing under P202's proposal.

What of P205? He is not so fortunate! He cannot count on the votes of P203 or P204: if they vote against him, they will have the fun of throwing him overboard and can still save themselves. So P205 gets thrown overboard no matter what he proposes. So does P206—he can be sure of P205's vote, but that's not enough. Similarly, P207 needs 104 votes—three plus his own plus 100 from bribes. He can get the votes of P205 and P206, but he needs one more, and it's not available. So P207 also walks the plank.

P208 is more fortunate. He also needs 104 votes, but P205, P206 and P207 will vote for him! Add in his own vote and 100 bribes, and he's in business. The recipients of his bribes must be among those who would definitely get nothing under P204's proposal: the even-numbered pirates P2 through P200, P201, P203 and P204.

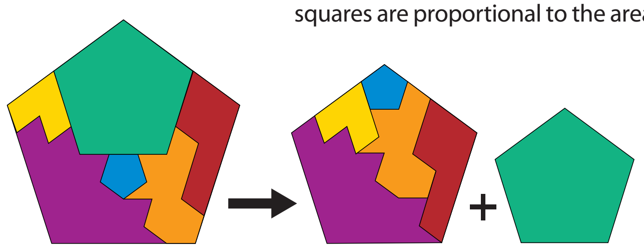
Now a new pattern has set in, and it continues indefinitely. Pirates who can make winning proposals (always to give themselves nothing and to bribe 100 fellow pirates) are separated from one another by ever longer sequences of pirates who will be thrown overboard no matter what proposal they make—and whose vote is therefore ensured for any fiercer pirate's proposal. The pirates who avoid this fate are P201, P202, P204, P208, P216, P232, P264, P328, P456 and so on—pirates whose number equals 200 plus a power of 2.

We must now decide which pirates are the lucky recipients of the bribes, just to make sure they will accept them. As I said, the solution is not unique, but one way to do this is for P201 to offer bribes to the odd-numbered pirates P1 through P199, for P202 to offer bribes to the even-numbered pirates P2 through P200, then P204 to the odd numbers, P208 to the evens and so on, alternating from odd to even and back again.

We conclude that with 500 pirates and optimal strategy, the first 44 pirates are thrown overboard, and then P456 offers one gold piece to each of the odd-numbered pirates P1 through P199. Thanks to their democratic system, the pirates have arranged their affairs so that the very fierce ones mostly get thrown overboard and can consider themselves lucky to escape death with none of the loot. Only the 200 meekest pirates have a chance of getting anything, and only half of *them* will actually receive a gold piece. Truly, the meek shall inherit the worth. SA

FEEDBACK

In response to the column on dissections ["Two-Way Jigsaw Puzzles," October 1997], Gil Lamb of the Principality of Andorra sent me a pentagon dissection (*below*) that provides an ingenious proof of the Pythagorean relation $3^2 + 4^2 = 5^2$. The numbers represent the lengths of the sides of the pentagons, so their squares are proportional to the areas. —*I.S.*



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